THE CONJUGATE PROBLEM OF HEAT TRANSFER AND THE OPTIMAL CONTROL OF NONSTATIONARY HEAT PROCESSES IN A NUCLEAR REACTOR

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The conditions are determined for optimal control in the conjugate one-dimensional heat-transfer problem for a water-cooled water-moderated nuclear reactor.

From the physical point of view the most complete system of equations describing nonstationary heat transfer between a fuel element of a nuclear reactor and the coolant flowing past it is the system of equations which includes not only the nonstationary thermal conductivity equation for the material of the fuel element and the equation for the convective heat transfer to the coolant, but also the hydrodynamical equation. But the numerical realization of the nonstationary conjugate problem gives rise to great difficulties. To simplify the "conjugate" system of equations we can avoid introducing the equation of hydrodynamics [1]. This is because for incompressible media it is possible to determine the velocity distribution in the coolant medium from the heat balance at the surface of the fuel element without solving the hydrodynamical boundary-value problem. In addition, the velocity distribution in the coolant can be assumed to be independent of the time. This observation is linked with the fact that in the transient process the thermal unsteadiness in the flow of the coolant past the fuel element is always greatly retarded by comparison with the onset of steady hydrodynamical flow.

Thus the problem of solving the nonstationary problem of optimal control of the thermal process in a nuclear reactor can be reduced to the analysis, using the principle of the maximum, of the conjugate boundary-value problem which includes a parabolic equation for the thermal conductivity and a hyperbolic equation for convective heat transfer in the coolant. Analytic methods of solving similar stationary conjugate problems were first developed in [2, 3]. At the same time we note that in many practical cases the original system of equations for analyzing the control of the nonstationary heat process in a nuclear reactor can be described in a simpler form if we replace the heat distribution function for the material of the fuel element by a stationary [4] or quasistationary [5] temperature distribution at its surface.

In spite of what has been said above, the general formulation of this class of physical problems for the control of the nonstationary heat process in a nuclear reactor [6] remains unchanged: by controlling the coolant velocity and the reactivity, we optimize the heat extraction from the fuel element surface, obtaining the maximal permissible heat extraction when the coolant temperature at the end of the fuel element does not exceed given values. Below we propose to assume that the fuel element and coolant temperatures vary along the length of the fuel element. This case is particularly interesting for water-moderated, water-cooled reactors in which thin cylindrical fuel elements with a length to radius ratio of nearly 500 are used. The convective heat-transfer coefficient along the coolant column in the channel in which fuel elements are axisymmetrically placed is taken to be constant.

Finally in the case of cylindrical geometry (Fig. 1) the conjugate problem is formulated as follows.

The parabolic thermal conductivity equation for the material of the thin cylindrical fuel elements with internal heat sources has the form

$$\frac{\partial t(z,\ \tau)}{\partial \tau} = a\,\frac{\partial^2 t(z,\ \tau)}{\partial z^2} + \frac{Q(\tau)}{c_0\gamma_0}\,J_0(k\ R_1)\sin\frac{\pi}{L'}(z+d) - \frac{\alpha' P_0}{c_0\gamma_0 F_f}\,[t(z,\ \tau) - \Theta(z,\ \tau)]. \tag{1}$$

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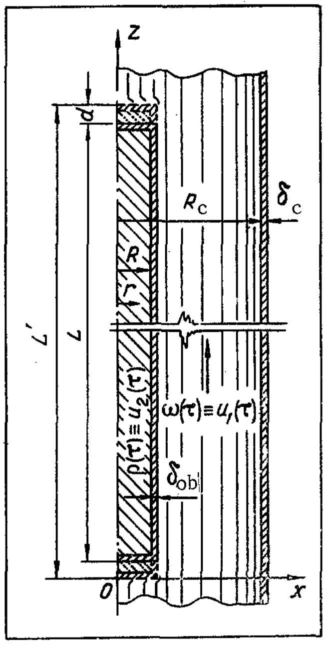

Fig. 1. Geometry of the conjugate system.

The hyperbolic heat balance equation at the fuel element surface can be written as

$$2\pi R_1\alpha' [t(z,\ \tau)-\theta(z,\ \tau)] = c_1\gamma_1 F_c\left(\frac{\partial\theta(z,\ \tau)}{\partial\tau}\right)+w(\tau)\frac{\partial\theta(z,\ \tau)}{\partial z}\Bigg), \quad (2)$$

where the functions $t(z, \tau)$ and $\theta(z, \tau)$ in Eqs. (1) and (2) satisfy the following boundary conditions:

$$\begin{gathered} t(z,\ \tau)|_{z=0}=0, \\ \theta(z,\ \tau)|_{z=0}=\text{const}, \\ \left.\frac{\partial t(z,\ \tau)}{\partial z}\right|_{z=L'}=\left.\frac{\partial\theta(z,\ \tau)}{\partial z}\right|_{z=L'}=0, \end{gathered} \quad (3)$$

where

$$0\leqslant z\leqslant L',$$

$$0\leqslant\tau\leqslant T,$$

$$L'=L+2d,$$

and the following initial conditions:

$$t(z,\ \tau)|_{\tau=0}=\frac{Q_0F_f}{P_0\ a}\sin\frac{\pi}{L'}(z+d)+\frac{Q_0F_fL}{\pi F_c w(0)\ c_1\gamma_1}\left[1-\cos\frac{\pi}{L'}(z+d)\right]+\theta(z,\ \tau)|_{z=0}; \quad (4)$$

$$\theta(z,\ \tau)|_{\tau=0}=\frac{Q_0F_fL}{\pi F_c w(0)\ c_1\gamma_1}\left[1-\cos\frac{\pi}{E'}(z+d)\right]+\theta(z,\tau)|_{z=0}.$$

We introduce the criterial functions:

$$\text{a)}\ J_1=2\pi R_1\alpha'\int\limits_0^{L'}\int^{T}[t(z,\ \tau)]-\theta(z,\ \tau)dzd\tau=\max, \quad (5)$$

describing the maximum heat extraction from the fuel element surface:

$$\text{b)}\ J_2=\int\limits_0^{T}N/2\,[\Theta(L',\ \tau)-[\Theta]]^2d\tau\leqslant[\varepsilon], \quad (6)$$

defining the permissible deviation of the coolant temperature from the given value at the end of the channel.

The form of the time function of the source $Q(\tau)$ is found by solving the familiar neutron diffusion equation and for a cylindrical fuel element we can write [7]:

$$Q(\tau)=Q_0\left(\frac{1}{1-\rho(\tau)/\beta}\exp\left[\frac{\lambda\rho(\tau)\tau}{\beta-\rho(\tau)}\right]-\frac{\rho(\tau)/\beta}{1-\rho(\tau)/\beta}\exp\left[-\left(\frac{\beta-\rho(\tau)}{l}\right)\tau\right]\right). \quad (7)$$

In our problem, we assume that the time functions for the coolant velocity $w(\tau)$ and the reactivity $\rho(\tau)$ permit goal-directed changes and can be taken as the control functions. We denote them respectively by the new variables $u_1(\tau)$ and $u_2(\tau)$. We note also that, in view of the physical conditions, $u_1(\tau)$ and $u_2(\tau)$ satisfy the following constraints:

$$\begin{gathered} u_{1\min}\leqslant u_1(\tau)\leqslant u_{1\max}, \\ u_{2\min}\leqslant u_2(\tau)\leqslant u_{2\max}. \end{gathered} \quad (8)$$

To solve (1) and (2) numerically we use the method of straight lines with the following approximate expressions:

$$\begin{gathered} \left.\frac{\partial^2 t(z,\ \tau)}{\partial z^2}\right|_{z=z_i}=\frac{1}{(h_z)^2}(t_{i+1}-2t_i+t_{i-1}), \\ \left.\frac{\partial\,\theta(z,\ \tau)}{\partial z}\right|_{z=z_i}=\frac{1}{2h_z}(\theta_{i+1}-\theta_{i-1}), \end{gathered} \quad (9)$$

where

$$t_i = t(z_i, \tau); \ \theta_i = \theta(z_i, \tau). \ z_i = h_z i, \ i = 1, 2, \ldots, n;$$

n is the number of approximating straight lines along the z-axis with step h_z.

We introduce the new variables:

$$\begin{gathered} x_1 \equiv 2\pi R_1 \alpha' \int_0^\tau \sum h_z (t_i - \theta_i)\, d\tau; \\ x_{i+1} \equiv t_i(\tau); \ x_{i+n+1} \equiv \theta_i(\tau) \ (i = 1, 2, \ldots, n); \\ x_{2n+2} \equiv \int_0^\tau \frac{N}{2} [\theta_n(\tau) - [\theta]]^2 d\tau. \end{gathered} \tag{10}$$

As a result of the approximation of the partial derivatives (9), and using the new variables (10) in Eqs. (1) and (2), we obtain a system of equations which is convenient for the application of the principle of the maximum [8]:

$$\begin{aligned} \dot{x}_1 &= \alpha' 2\pi R_1 h_z \sum_{i=1}^{n} (x_{i+1} - x_{i+n+1}); \\ \dot{x}_2 &= \frac{a}{h_z^2}(x_3 - 2x_2) + \frac{1}{c_0 \gamma_0}[Q(z_1, u_2) - Q^*(x_2, x_{2+n})]; \\ \dot{x}_{i+1} &= \frac{a}{h_z^2}(x_{i+2} - 2x_{i+1} + x_i) + \frac{1}{c_0\gamma_0}[Q(z_i, u_2) - Q^*(x_{i+1}, x_{i+n+1})] \\ & \qquad i = 2, 3, \ldots, n-1; \\ \dot{x}_{n+1} &= \frac{a}{h_z^2}(x_n - x_{n+1}) + \frac{1}{c_0\gamma_0}[Q(z_n, u_2) - Q^*(x_{n+1}, x_{2n+1})]; \\ \dot{x}_{n+2} &= \frac{2\pi R_1 \alpha'}{c_1 \gamma_1 F_c}(x_2 - x_{n+2}) - u_1 \frac{1}{2h_z}(x_{n+3} - 0); \\ \dot{x}_{i+n+1} &= \frac{2\pi R_1 \alpha'}{c_1 \gamma_1 F_c}(x_{i+1} - x_{i+n+1}) - u_1 \frac{1}{2h_z}(x_{i+n+2} - x_{i+n}) \\ & \qquad (i = 2, 3, \ldots, n-1); \\ \dot{x}_{2n+1} &= \frac{2\pi R_1 \alpha'}{c_1 \gamma_1 F_c}(x_{n+1} - x_{2n+1}) - u_1 \frac{1}{2h_z}(x_{2n+1} - x_{2n}); \\ \dot{x}_{2n+2} &= N/2\,(x_{2n+1} - [\theta])^2, \end{aligned} \tag{11}$$

where

$$Q(z_i, u_2) = Q_0 \sin\frac{\pi}{L'}(z_i + d)\left(\frac{\beta}{\beta - u_2}\exp\left[\frac{\lambda u_2 \tau}{\beta - u_2}\right] - \frac{u_2}{\beta - u_2}\exp\left[-\frac{\beta - u_2}{l}\tau\right]\right);$$

$$Q^*(x_{i+1}, x_{i+n+1}) = \frac{2\alpha'}{R_1}(x_{i+1} - x_{i+n+1}) \ (i = 1, 2, \ldots, n).$$

The initial conditions for Eqs. (11) are

$$\begin{aligned} & x_1(0) = 0; \\ & x_{i+1}(0) = \frac{Q_0 F_f}{P_0 a} \sin\frac{\pi}{L'}(z + d) + \frac{Q_0 F_f L}{\pi F_c w(0) c_1 \gamma_1} \\ & \qquad \times \left[1 - \cos\frac{\pi}{L'}(z + d)\right] + \theta(z, \tau)|_{z=0} \\ & \qquad (i = 1, 2, \ldots, n); \\ & x_{i+n+1}(0) = \frac{Q_0 F_f L}{\pi w(0) \gamma_1 c_1 F_c}\left[1 - \cos\frac{\pi}{L'}(z_i + d)\right] + \theta(z, \tau)_{z=0} \\ & \qquad (i = 1, 2, \ldots, n); \\ & x_{2n+2}(0) = 0. \end{aligned} \tag{12}$$

In addition, the functional to be maximized can be written as

$$J_1 = x_1(\tau),$$

and we can write the necessary condition as

$$x_{2n+2} \leqslant [\varepsilon]. \tag{13}$$

The value [ε] is chosen from the dynamical conditions under which the whole nuclear reactor operates.

Thus, in the new notation the mathematical formulation has been reduced to the following. It is required to find control functions $u_1(\tau)$ and $u_2(\tau)$ subject to the constraints (8) and the corresponding functions $x_i^{(0)}(T)$ (i = 1, 2, . . . , n) such that the heat extraction from the fuel element surface by the coolant is maximal under the assumption that condition (6) holds. Following the principle of the maximum we introduce the function

$$H(x_1,\ x_2,\ \ldots,\ x_n;\ \psi_1,\ \psi_2,\ \ldots,\ \psi_n;\ u_1;\ u_2;\ \tau),$$

which explicitly is

$$\begin{aligned} H(x,\ \psi,\ u,\ \tau) = &\left[\alpha' 2\pi R_1 h_z \sum_{i=1}^{n} (x_{i+1} - x_{i+n+1})\right]\psi_1 \\ &+ \left\{\frac{a}{h_z^2}(x_3 - 2x_2) + \frac{1}{c_0\gamma_0}[Q(z_i,\ u_2) - Q^*(x_2,\ x_{2+n})]\right\}\psi_2 + \sum_{i=2}^{n-1}\left\{\frac{a}{h_z^2}(x_{i+2} - 2x_{i+1} + x_i) + \frac{1}{c_0\gamma_0}[Q(z_i,\ u_2)\right. \\ &\left. - Q^*(x_{i+1},\ x_{i+n+1})]\right\}\psi_{i+1} + \left\{\frac{a}{h_z^2}(x_n - x_{n+1}) + \frac{1}{c_0\gamma_0}[Q(z_n,\ u_2) - Q^*(x_{n+1},\ x_{2n+1})]\right\}\psi_{n+1} + \left[\frac{2\pi R_1\alpha'}{c_1\gamma_1 F_c}(x_2 - x_{2n+2})\right. \\ &\left. - u_1\frac{1}{2h_z}x_{n+3}\right]\psi_{n+2} + \sum_{i=2}^{n-1}\left[\frac{2\pi R_1\alpha'}{c_1\gamma_1 F_c}(x_{i+1} - x_{i+n+1}) - u_1\frac{1}{2h_z}\right. \\ &\left. \times (x_{i+n+2} - x_{i+n})\right]\psi_{i+n+1} + \left[\frac{2\pi R_1\alpha'}{c_1\gamma_1 F_c}(x_{n+1} - x_{2n+1}) - u_1\frac{1}{2h_z}\right. \\ &\left. \times (x_{2n+1} - x_{2n})\right]\psi_{2n+1} + [N/2\,(x_{2n+1} - [\theta])^2]\psi_{2n+2}, \end{aligned} \tag{14}$$

where the $\psi_1, \psi_2, \ldots, \psi_{2n+2}$ are conjugate functions satisfying the following system of equations:

$$\begin{aligned} &\dot{\psi}_1 = 0; \\ &\dot{\psi}_2 = -\alpha' 2\pi R_1 h_z \psi_1 + \left(\frac{2a}{h_z^2} + \frac{2\alpha'}{c_0\gamma_0 R_1}\right)\psi_2 - \frac{a}{h_z^2}\psi_3 - \frac{2\pi R_1\alpha'}{c_1\gamma_1 F_c}\psi_{n+2}; \\ &\dot{\psi}_{i+1} = -\alpha' 2\pi R_1 h_z \psi_1 + \left(\frac{2a}{h_z^2} + \frac{2\alpha'}{c_0\gamma_0 R_1}\right)\psi_{i+1} - \frac{a}{h_z^2}(\psi_i + \psi_{i+2}) - \frac{2\pi R_1\alpha'}{c_1\gamma_1 F_c}\psi_{i+n+1} \\ &\qquad (i = 2,\ 3,\ \ldots,\ (n-1)); \\ &\dot{\psi}_{n+1} = -\alpha' 2\pi R_1 h_z \psi_1 + \left(\frac{a}{h_z^2} + \frac{2\alpha'}{c_0\gamma_0 R_1}\right)\psi_{n+1} - \frac{a}{h_z^2}\psi_n - \frac{2\pi R_1\alpha'}{c_1\gamma_1 F_c}\psi_{2n+1}; \\ &\dot{\psi}_{i+n+1} = \alpha' 2\pi R_1 h_z \psi_1 - \frac{2\alpha'}{c_0\gamma_0 R_1}\psi_{i+1} + \frac{2\pi R_1\alpha'}{c_1\gamma_1 F_c}\psi_{i+n+1} + \frac{u_1}{2h_z}\psi_{i+n} - \frac{u_1}{2h_z}\psi_{i+n+2} \\ &\qquad (i = 2,\ 3,\ \ldots,\ (n-1)); \\ &\dot{\psi}_{2n+1} = -\alpha' 2\pi R_1 h_z \psi_1 - \frac{2\alpha'}{c_0\gamma_0 R_1}\psi_{n+1} + \frac{2\pi R_1\alpha'}{c_1\gamma_1 F_c}\psi_{2n+1} + \frac{u_1}{2h_z}\psi_{2n+1} - N(x_{2n+1} - [\theta])\psi_{2n+2}; \\ &\dot{\psi}_{2n+2} = 0; \end{aligned} \tag{15}$$

the boundary conditions for (15) are:

$$\begin{aligned} &\psi_1(T) = -1; \\ &\psi_2(T) = \psi_3(T) = \ldots = \psi_{2n+2}(T) = 0. \end{aligned} \tag{16}$$

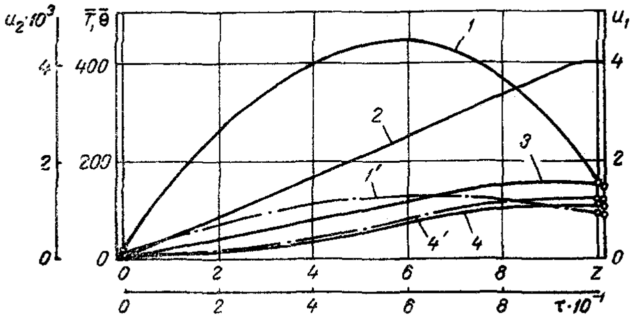

Fig. 2

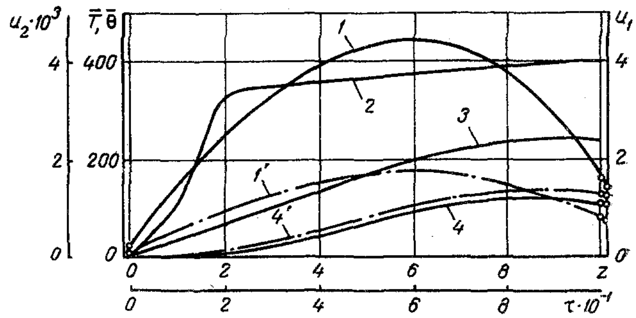

Fig. 3

Fig. 2. Linear nature of the change in the control function $u_2(\tau)$ (2) and the corresponding temperature distribution (°C) at the fuel element surface $t(R, z, \tau)$ (1), (1') and in the coolant $\theta(z, \tau)$ (4), (4'): 1, 4) $\alpha' = 0.1$; 1', 4') $\alpha' = 1$ cal/cm^2·sec·deg; 3) $u_1(\tau)$ – the control function for the coolant velocity.

Fig. 3. Effect of the chosen criteria on the nature of the changes in the control functions $u_1(\tau)$ (3) and $u_2(\tau)$ (2) and the corresponding temperature distributions (°C) at the fuel element surface $t(R, z, \tau)$ (1), (1') and in the coolant $\theta(z, \tau)$ (4), (4'): 1, 4) $\alpha' = 0.1$; 1', 4') $\alpha' = 1$ cal/cm^2·sec·deg.

As we see, the problem is typical of problems in optimal control with a free right end. To determine the optimal controls and the corresponding optimal trajectories we have to find the extremum of the function

$$H(x_1, x_2 \ldots, x_n; \psi_1, \psi_2 \ldots, \psi_n; u_1, u_2, \tau)$$

taking account of the heat engineering constraint (6) for the given fuel element materials and the coolant.

The principle result of the analysis of the systems (11) and (15) is the determination of the optimal temperature distributions up the height of the fuel element and along the coolant layer flowing past the fuel element in accordance with the given initial thermal output of the source.

By analyzing the numerical results we obtain (Figs. 2 and 3) curve 2, $\rho(\tau)$, and curve 3, $w(\tau)$, defining two variants of optimal control of the thermal process in a water-moderated, water-cooled reactor in the transition period to a new output level. The initial fuel element output in both variants is the same, having the value 100 cal/cm^3·sec. The fundamental fuel element parameters for the numerical evaluation of the solution are taken from [9].

In the first variant (Fig. 2), a linear law is specified for the change in reactivity and the resulting change in the coolant velocity is nearly linear. In the second variant both control parameters vary arbitrarily within a given permissible region (Fig. 3).

The nature of the temperature distribution at the fuel element surface and in the coolant layer in both variants is described by the curves 1, 1' and 4, 4' for $\overline{T}(z, \tau)$ and $\overline{\theta}(z, \tau)$ respectively for two different reduced heat-transfer coefficients $\alpha_1' = 0.1$ and $\alpha_2' = 1$.

From the curves we see that when the change in reactivity is linear and the coolant velocity change is nearly linear, the values of the temperature at the fuel element surface and in the coolant layer are close to the values of the temperature obtained analytically [10] and correspond to the operational transient regime of the water-moderated, water-cooled reactor [11, 12].

The numerical analysis showed that:

1. For the operational equations we can take the conjugate parabolic–hyperbolic system of equations describing nonstationary heat transfer in a water-moderated, water-cooled nuclear reactor fuel element.

2. The effective control parameters are the time functions of the reactivity and the coolant velocity.

3. To construct approximate one-dimensional thermal fields along the fuel element surface it is convenient to assume that the convective heat-transfer coefficient lies within the limits 0.1–1.0 cal/cm^2·sec·deg.

4. The principle of the maximum defines (is the formal expression of) the optimality condition.

5. The computational algorithm we have constructed makes it possible to determine numerically both the nature of the changes in the control parameters which are being optimized and the temperature distributions at the fuel element surface and in the coolant layer when at the same time we have maximum heat extraction and minimum deviation of the temperature from the given value at the end of the channel of the active zone of the reactor during the transient process.

NOTATION

a is the thermal conductivity coefficient;
α' is the reduced heat-transfer coefficient;
c_0, γ_0 are the heat capacity and specific density of the fuel element material;
R_1 is the external radius of the fuel element;
c_1, γ_1 are the heat capacity and specific density of coolant;
$t(z, \tau)$ is the temperature distribution in fuel element material averaged over the cross section;
λ_f is the heat-conduction coefficient of fuel element material;
$\theta(z, \tau)$ is the temperature distribution in the coolant layer along the axis averaged over the channel cross section;
F_f is the cross-sectional area of fuel element;
F_c is the cross-sectional area of channel;
w is the coolant velocity;
$w(0)$ is the initial coolant velocity;
L' is the extrapolated fuel element length along the z-axis ($L' = L + 2d$);
d is the extrapolation length;
L is the geometrical fuel element length along the z-axis;
Q_0 is the initial source output;
z is the space variable along fuel element axis;
τ is the time;
ρ is the fuel element reactivity;
λ is the nuclear decay constant;
l is the mean neutron life time;
β is the fraction of delayed neutrons;
P_0 is the washed perimeter of the fuel element.

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